A small inequality problem

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This is a small inequality sent to me by Bowen Kwan to prove:

$$\sqrt[3]{(a_1+b_1)(a_2+b_2)(a_3+b_3)} \geq \sqrt[3]{a_1a_2a_3} + \sqrt[3]{b_1b_2b_3} \quad \text{where } a_i, b_i \geq 0 \qquad \text{....} \quad \textbf{(1)}$$

My last teaching day with 7E class. I acted as a substitute teacher for this vivacious and happy class after my retirement in 2009. (Photo taken in 2011)

My first attempt is to use brute force to cube the inequality:

$$\sqrt[3]{(a_1+b_1)(a_2+b_2)(a_3+b_3)} \geq \sqrt[3]{a_1a_2a_3} + \sqrt[3]{b_1b_2b_3}$$

$$\Leftrightarrow (a_1+b_1)(a_2+b_2)(a_3+b_3) \geq \left(\sqrt[3]{a_1a_2a_3} + \sqrt[3]{b_1b_2b_3}\right)^3$$

$$\Leftrightarrow (a_1a_2b_3 + a_2a_3b_1 + a_3a_1b_2) + (a_1b_2b_3 + a_2b_3b_1 + a_3b_1b_2)$$
$$\geq 3\left(\sqrt[3]{a_1a_2a_3}\right)^2\left(\sqrt[3]{b_1b_2b_3}\right) + 3\left(\sqrt[3]{a_1a_2a_3}\right)\left(\sqrt[3]{b_1b_2b_3}\right)^2$$

$$\Leftrightarrow \left(\frac{a_1a_2b_3+a_2a_3b_1+a_3a_1b_2}{3}\right) + \left(\frac{a_1b_2b_3+a_2b_3b_1+a_3b_1b_2}{3}\right)$$
$$\geq \sqrt[3]{(a_1a_2b_3)(a_2a_3b_1)(a_3a_1b_2)} + \sqrt[3]{(a_1b_2b_3)(a_2b_3b_1)(a_3b_1b_2)}$$

which is obviously true by using A.M. ≥ G.M. twice.

Since Bowen hinted that whether this inequality can be proved by CBS inequality, so I began my second attempt. Using the CBS inequality for two sets of variables case looked easy:

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) \geq (x_1x_2 + y_1y_2)^2$$

Putting $x_1^2 = a_1$, $y_1^2 = b_1$, $x_2^2 = a_2$, $y_2^2 = b_2$, we get:

$$(a_1+b_1)(a_2+b_2) \geq \left(\sqrt{a_1a_2} + \sqrt{b_1b_2}\right)^2$$

or $$\sqrt{(a_1+b_1)(a_2+b_2)} \geq \sqrt{a_1a_2} + \sqrt{b_1b_2} \qquad \text{....} \quad \textbf{(2)}$$

Although **(2)** is not we wanted, but it is a good step forwards to have my proof.

Applying the C.B.S. inequality directly is not fruitful for proving **(1)**, the reader may try.
It is lucky that we can extend to 4 sets of variables case:

$$\sqrt[4]{(a_1+b_1)(a_2+b_2)(a_3+b_3)(a_4+b_4)} = \sqrt{\sqrt{(a_1+b_1)(a_2+b_2)}\sqrt{(a_3+b_3)(a_4+b_4)}}$$

$$\geq \sqrt{\left(\sqrt{a_1a_2}+\sqrt{b_1b_2}\right)\left(\sqrt{a_3a_4}+\sqrt{b_3b_4}\right)}$$

$$\geq \sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}}+\sqrt{\sqrt{b_1b_2}\sqrt{b_3b_4}}$$

$$= \sqrt[4]{a_1a_2a_3a_4}+\sqrt[4]{b_1b_2b_3b_4} \qquad \text{.... (3)}$$

To get back to **(1)**, we need to put away a_4, b_4 .
We let $a_4 = \sqrt[3]{a_1a_2a_3}$, $b_4 = \sqrt[3]{b_1b_2b_3}$ and substitute in **(3)** .

$$\sqrt[4]{(a_1+b_1)(a_2+b_2)(a_3+b_3)(a_4+b_4)} \geq \sqrt[4]{a_1a_2a_3a_4}+\sqrt[4]{b_1b_2b_3b_4}$$

$$\sqrt[4]{(a_1+b_1)(a_2+b_2)(a_3+b_3)\left(\sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}\right)} \geq \sqrt[4]{a_1a_2a_3\sqrt[3]{a_1a_2a_3}}+\sqrt[4]{b_1b_2b_3\sqrt[3]{b_1b_2b_3}}$$

$$\sqrt[4]{(a_1+b_1)(a_2+b_2)(a_3+b_3)}\sqrt[4]{\left(\sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}\right)} \geq \left(\sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}\right)$$

$$\sqrt[4]{(a_1+b_1)(a_2+b_2)(a_3+b_3)} \geq \sqrt[4]{\left(\sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}\right)^3}$$

$$\sqrt[3]{(a_1+b_1)(a_2+b_2)(a_3+b_3)} \geq \sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}$$

Using CBS inequality to prove **(1)** is obviously tedious, but it can give a nice extension of this inequality as we shall see.

My third attempt to deal with (1) is given below. I use "think backwards, write forwards" and the reader may trace my way of thinking by beginning at the last line and read backwards.

By A.M. ≥ G.M.,

$$\frac{\frac{a_1}{a_1+b_1}+\frac{a_2}{a_2+b_2}+\frac{a_3}{a_3+b_3}}{3}+\frac{\frac{b_1}{a_1+b_1}+\frac{b_2}{a_2+b_2}+\frac{b_3}{a_3+b_3}}{3}$$

$$\geq \sqrt[3]{\frac{a_1}{a_1+b_1}\frac{a_2}{a_2+b_2}\frac{a_3}{a_3+b_3}}+\sqrt[3]{\frac{b_1}{a_1+b_1}\frac{b_2}{a_2+b_2}\frac{b_3}{a_3+b_3}}$$

$$\Leftrightarrow 1 \geq \frac{\sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}}{\sqrt[3]{(a_1+b_1)(a_2+b_2)(a_3+b_3)}}$$

$$\Leftrightarrow \sqrt[3]{(a_1+b_1)(a_2+b_2)(a_3+b_3)} \geq \sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}$$

Nice, short, isn't it?

Last, but not the least, I try to give **(1)** a possible extension.

By studying **(1)**, **(2)** and **(3)**, we can get:

$$\prod_{i=1}^{n}(a_i + b_i) \geq \sqrt[n]{\prod_{i=1}^{n} a_i} + \sqrt[n]{\prod_{i=1}^{n} b_i} \qquad \ldots . (\mathbf{4})$$

Two possible proofs of **(4)** are :

(a) Use "Backward mathematical induction". (a very long proof)

(b) Use my "third attempt". (a very short proof)

I leave the proofs for the reader as an exercise. For those who don't know much on backward induction may find my example useful:

http://www.qc.edu.hk/math/Resource/AL/AGM_1.pdf